HOSSAM GHANEM

(4) 7.3 The Logarithm Function

(1) $\boxed{a^x = y} \Leftrightarrow \boxed{x = \log_a y}$

Example	$5^2 = 25 \Leftrightarrow \log_5 25 = 2$	$10^3 = 1000 \Leftrightarrow \log_{10} 1000 = 3$

(2) $\boxed{y = e^x} \Leftrightarrow \boxed{\log_e y = x} \Leftrightarrow \boxed{\ln y = x}$

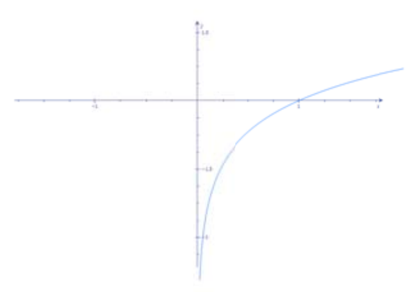
(3) $\boxed{\ln e^x = x}$, $\boxed{e^{\ln x} = x}$

$\ln e = 1$	$\ln 1 = 0$
$\ln 0$ Not definite	$\ln x \; , x < 0$ Not definite

(1) $\log_a x = \dfrac{\ln x}{\ln a}$	$\log_3 x = \dfrac{\ln x}{\ln 3}$
(2) $\ln AB = \ln A + \ln B$	$\ln(x^2 - 8)(x + 5) = \ln(x^2 - 8) + \ln(x + 5)$
(3) $\ln\left(\dfrac{A}{B}\right) = \ln A - \ln B$	$\ln \dfrac{(2x^3 - 16)}{\cos x} = \ln(2x^3 - 16) - \ln \cos x$
(4) $\ln x^r = r \ln x$	$\ln x^{\sqrt{x}} = \sqrt{x} \ln x$

Example

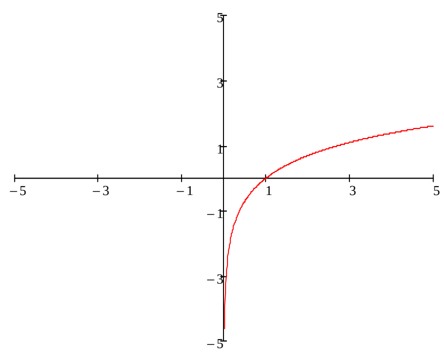
$$\ln\left(\frac{\sqrt[3]{x+1}\,\sec x}{\sqrt{x}\,\sin x}\right) = \frac{1}{3}\ln(x+1) + \ln \sec x - \frac{1}{2}\ln x - \ln \sin x$$



$y = \log_a x$

Graph

$\log_e x = \ln x$



$y = \ln x$

Domain:

$$f(x) = \ln x \quad \Rightarrow \quad D_f = (\ 0\ , \infty) \quad , \quad R_f = \mathcal{R}$$

Limit:

If $a > 0$, $\lim_{x \to a} (\ln x) = \ln a$	$\lim_{x \to 3} (\ln x) = \ln 3$
If $a < 0$, $\lim_{x \to a} (\ln x)$ $D.N.E$	$\lim_{x \to -6} (\ln x)$ $D.N.E$

$\lim_{x \to 0^+} (\ln x) = -\infty$	$\lim_{x \to 0^-} (\ln x)$ $D.N.E$	$\lim_{x \to 0} (\ln x)$ $D.N.E$

$\lim_{x \to \infty} (\ln x) = \infty$	$\lim_{x \to -\infty} (\ln x)$ $D.N.E$

Differential :

$\frac{d}{dx} \log_a(x) = \frac{1}{\ln a} \cdot \frac{1}{x}$	$\frac{d}{dx} \log_9(x) = \frac{1}{\ln 9} \cdot \frac{1}{x}$
$\frac{d}{dx} \ln x = \frac{1}{x}$	$\frac{d}{dx} \ln(x-5) = \frac{1}{x-5}$

Notes

$1 = \log_a a$	$2 = \log_a a^2$	$3 = \log_a a^3$	$n = \log_a a^n$
$1 = \log_6 6$	$2 = \log_4 16$	$3 = \log_2 8$	$3 = \log_2 2^3 = \log_2 8$
$1 = \log_5 5$	$2 = \log_7 49$	$3 = \log_3 27$	$3 = \log_5 5^3 = \log_5 125$
$1 = \ln e$ ⇓ $1 = \log_n n$	$2 = \ln e^2$ ⇓ $2 = \log_n n^2$	$3 = \ln e^3$ ⇓ $3 = \log_n n^3$	$4 = \log_2 2^4 = \log_2 16$ $4 = \log_3 3^4 = \log_3 234$

$$\log_a x = \frac{1}{\ln a} \ln x$$

$$\log_x y \ \log_y x = 1$$

Example 1

26 January 2006 A
32 Oct. 31st , 2010 A

Prove the identity: $\log_x(xy)\log_y(xy) = \log_x y + \log_y x + 2$, for any positive $x, y \neq 1$

Solution

$$L.H.S = \log_x(xy)\log_y(xy) = [\log_x x + \log_x y][\log_y x + \log_y y]$$
$$= [1 + \log_x y][\log_y x + 1] = \log_y x + 1 + \log_x y \log_y x + \log_x y$$
$$\because \log_x y \ \log_y x = \frac{\ln y}{\ln x} \cdot \frac{\ln x}{\ln y} = 1$$
$$\therefore L.H.S = \log_y x + 1 + 1 + \log_x y = \log_y x + \log_x y + 2 = R.H.S$$

Example 2

13 March 2001 A

Show that: $\operatorname{Log}_{x^2} x + \log_x x^2 = \frac{5}{2}$, for $x > 0$, $x \neq 1$

Solution

$$L.H.S = \log_{x^2} x + \log_x x^2 = \frac{\ln x}{\ln x^2} + \frac{\ln x^2}{\ln x} = \frac{\ln x}{2\ln x} + \frac{2\ln x}{\ln x} = \frac{1}{2} + 2 = \frac{5}{2} = R.H.S$$

Example 3

Show that $\log_2 x = 3\log_8 x$ for all $x > 0.$ 26 July 2008 A

Solution

$$L.H.S = \log_2 x = \frac{\ln x}{\ln 2}$$
$$R.H.S = 3\log_8 x = \frac{3\ln x}{\ln 8} = \frac{3\ln x}{\ln 2^3} = \frac{3\ln x}{3\ln 2} = \frac{\ln x}{\ln 2} = L.H.S$$

Example 4

27 Nov. 2008 A

Let $a > 1$. Solve the following equation for x. $\log_a(x+2) = \frac{2\ln x}{\ln a}$

Solution

$$\log_a(x+2) = \frac{2\ln x}{\ln a}$$
$$\frac{\ln(x+2)}{\ln a} = \frac{\ln x^2}{\ln a}$$
$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2$$

Example 5

19 March 2006 A

Find all solutions for the equation: $\log_6 x + \log_6(x-5) = 1$

Solution

$$\operatorname{Log}_6 x + \log_6(x-5) = 1$$
$$\operatorname{Log}_6 x\ (x-5) = \operatorname{Log}_6 6$$
$$x(x-5) = 6$$
$$x^2 - 5x - 6 = 0 \quad \rightarrow \quad (x-6)(x+1) = 0 \quad \rightarrow \quad x = 6$$

Example 6

Solve $\log_6(4-x) = 1 - \log_6(x+3)$ — 29 July 2009 A

Solution

$$\log_6(4-x) = 1 - \log_6(x+3)$$
$$\log_6(4-x) = \log_6 6 - \log_6(x+3)$$
$$\log_6(4-x) + \log_6(x+3) = \log_6 6$$
$$(4-x)(x+3) = 6$$
$$4x + 12 - x^2 - 3x = 6$$
$$-x^2 + x + 12 - 6 = 6$$
$$-x^2 + x + 6 = 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3 \quad \text{or} \quad x = -2$$

Example 7

Solve for x $\log_{\frac{1}{e}}(4-3x) < 1$ — 28 April 2009 A

Solution

$$\log_{\frac{1}{e}}(4-3x) < 1$$
$$\frac{\ln(4-3x)}{\ln\frac{1}{e}} < 1$$
$$\frac{\ln(4-3x)}{-\ln e} < 1$$
$$\ln(4-3x) > -\ln e$$
$$\ln(4-3x) > \ln\frac{1}{e}$$
$$4-3x > \frac{1}{e}$$
$$-3x > \frac{1}{e} - 4$$
$$x < \frac{\frac{1}{e} - 4}{-3}$$
$$x < \frac{1-4e}{-3e}$$
$$x < \frac{4e-1}{3e}$$

Example 8

Determine all x for which $\log_{\frac{1}{2}}(x^2-3) < 0$ — 24 May 2005 A

Solution

$$\log_{\frac{1}{2}}(x^2-3) < 0$$
$$\frac{\ln(x^2-3)}{\ln\frac{1}{2}} < 0$$
$$\frac{\ln(x^2-3)}{-\ln 2} < 0$$
$$\ln(x^2-3) > 0$$
$$\ln(x^2-3) > \ln 1$$
$$x^2 - 3 > 1$$
$$x^2 > 4$$
$$|x| > 2$$
$$x < -2 \quad \text{or} \quad x > 2$$
$$x \in (-\infty, -2) \cup (2, \infty)$$
$$\text{or} \quad x \in R/[-2,2]$$

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Homework

1	Find all solutions for equation $\log_2 x + \log_2(x-6) = 4$	
2	Find all solutions for equation $\log_{x^2} 250 - \log_{x^2} 2 = \frac{3}{2}$	
3	Show that $\log_{x^3} x + \log_x x^3 = \frac{10}{3}$	
4	statements is true or false-explain your answer. $\ln\sqrt{a+b} = \frac{1}{2}(\ln a + \ln b)$	January 2005 A
5	(3 pts) Find all x for which $\log_2(x+1) = 1 + \log_{1/2}(x+2)$	30 April 11, 2010
6	(2 pts) Solve the equation $\log_2(x^2+1) - 2\log_2 x = 1$	33 April 10, 2011
7	(1.5 pts) . Answer the following as true or False . $\log_{1/3} 10 > 0$	33 April 10, 2011
8	Find the solution set of the inequality $\log_{\frac{1}{2}}(x^2-3) > 0$ (2 pts.)	35 January 24, 2010
9	(2 pts.) Prove that $\log_{1/x}(x) + \log_x(1/x) = -2$ for all $x > 1$.	36 June 6, 2010
10	(3 pts.) Prove that $\tanh(\log_3 x) = \dfrac{x^{2/\ln 3} - 1}{x^{2/\ln 3} + 1}$	38 Jan. 22, 2011